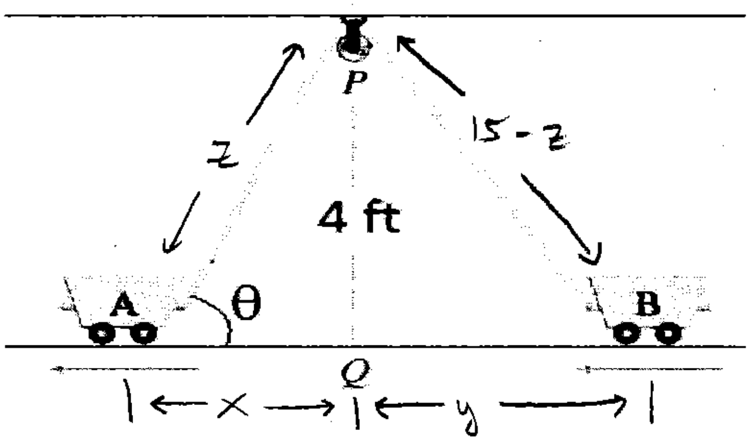
Closing *Thurs*: 3.9
Closing next *Tues*: 3.10
Closing next *Thurs*: 4.1(1), 4.1(2)
Remember: Friday is a Holiday (no class)

Entry Task (2016 Midterm Question):
Two carts are connected by a rope 15 ft long that passes over a pulley P. The point Q is on the floor 4 ft beneath P. Cart A is being pulled away from Q at a constant speed of 2 ft/s.

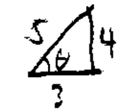
(a) Find the rate at which θ is changing when cart A is 3 ft from Q.

(b) How fast is cart B moving when cart A is 3 ft from Q?



(a) $\boxed{\text{KNOW}}$ $\frac{dx}{dt} = 2$

$\boxed{\text{WANT}}$ $\frac{d\theta}{dt} = ??$ WHEN $x = 3$



$\tan\theta = \frac{4}{x} \Rightarrow \sec^2\theta \frac{d\theta}{dt} = -4x^{-2}\frac{dx}{dt}$

PLUG IN

$\tan\theta = 4/5 \Rightarrow \sec\theta = 5/3$

$\Rightarrow \left(\frac{5}{3}\right)^2 \cdot \frac{d\theta}{dt} = -\frac{4}{(3)^2}(2)$

$\Rightarrow \boxed{\frac{d\theta}{dt} = -\frac{8}{9}\cdot\frac{9}{25} = -\frac{8}{25}\ \frac{\text{RAD}}{\text{SEC}}}$

(b) $\boxed{\text{WANT}}$ $\frac{dy}{dt} = ??$ WHEN $x = 3$

$x^2 + 4^2 = z^2 \Rightarrow \cancel{2}x\frac{dx}{dt} = \cancel{2}z\frac{dz}{dt}$

$y^2 + 4^2 = (15-z)^2 \Rightarrow \cancel{2}y\frac{dy}{dt} = \cancel{2}(15-z)(-1)\frac{dz}{dt}$

PLUG IN

• $x\frac{dx}{dt} = z\frac{dz}{dt} \Rightarrow (3)(2) = (5)\frac{dz}{dt} \Rightarrow \frac{dz}{dt} = \frac{6}{5}$

• $y^2 + 4^2 = (15-5)^2 \Rightarrow y^2 = 100 - 16 = 84$
$\Rightarrow y = \sqrt{84} = 2\sqrt{21}$

• $y\frac{dy}{dt} = -(15-z)\frac{dz}{dt} \Rightarrow 2\sqrt{21}\frac{dy}{dt} = -(15-5)\left(\frac{6}{5}\right) = -12$

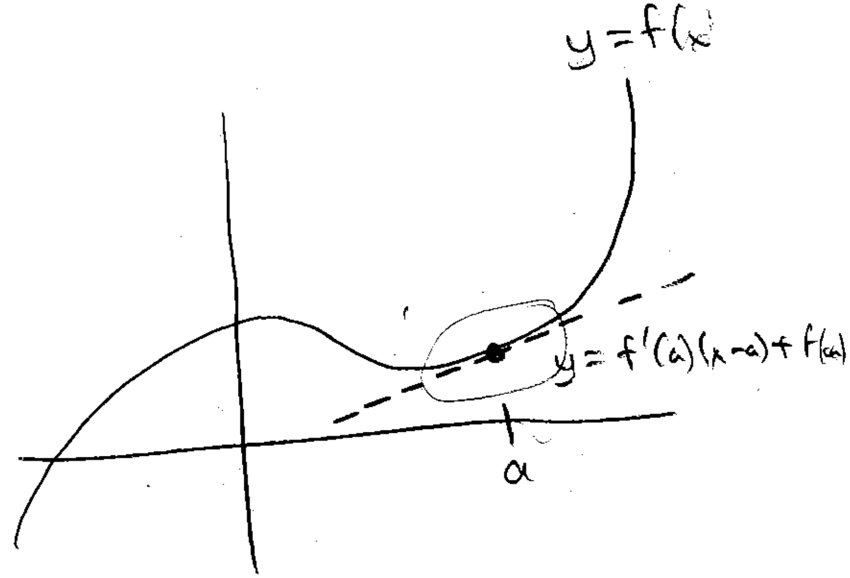
$\Rightarrow \boxed{\frac{dy}{dt} = -\frac{12}{2\sqrt{21}} = -\frac{6}{\sqrt{21}}\ \frac{\text{ft}}{\text{sec}}}$

3.10 Linear Approximation

Idea: "Near" the point $(a, f(a))$ the graphs of $y = f(x)$ and the tangent line

$$y = f'(a)(x - a) + f(a)$$

are close together.



$$f(x) \approx f'(a)(x-a) + f(a)$$

We say the tangent line is a **linear approximation** (or **linearization** or **tangent line approximation**) to the function. Sometimes it is written as

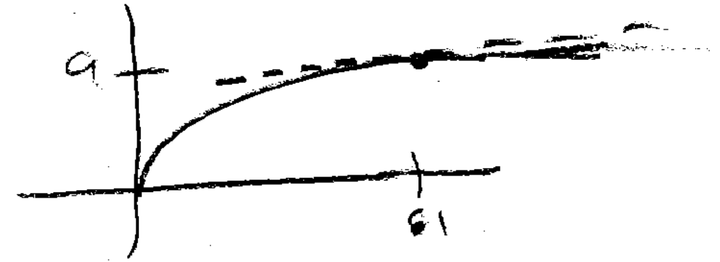
$$L(x) = f'(a)(x - a) + f(a)$$

In other words:

If $x \approx a$,

then $f(x) \approx f'(a)(x - a) + f(a)$

Example: Find the linear approximation of $f(x) = \sqrt{x}$ at $x = 81$. Then use it to approximate the value of $\sqrt{82}$.



$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$f(81) = \sqrt{81} = 9 = y\text{-VALUE}$$

$$f'(81) = \frac{1}{2\sqrt{81}} = \frac{1}{18} = \text{slope}$$

$$\boxed{\begin{aligned} y &= \frac{1}{18}(x-81) + 9 \\ L(x) &= \frac{1}{18}(x-81) + 9 \end{aligned}}$$

$$\sqrt{x} \approx \frac{1}{18}(x-81) + 9$$

FOR $x \approx 81$

THUS,

$$\underbrace{\sqrt{82}}_{9.055385138\ldots} \approx \underbrace{\frac{1}{18}(82-81) + 9}_{9.05\overline{5}} = 9 + \frac{1}{18}$$

Example: Find the linearization of $g(x) = \sin(x)$ at $x = 0$. Then use it to approximate the value of sin(0.03).

$g'(x) = \cos(x)$

$g(0) = \sin(0) = 0 = y\text{-VALUE}$

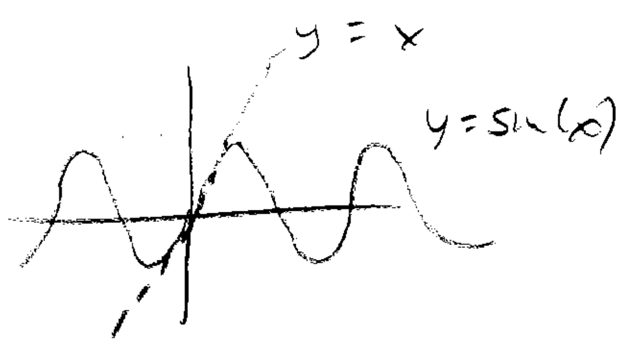
$g'(0) = \cos(0) = 1 = \text{slope}$

$y = 1(x-0) + 0$

$y = x$

$L(x) = x$

$$\boxed{\sin(x) \approx x \quad \text{for } x \approx 0}$$



THUS,

$$\underbrace{\sin(0.03)}_{0.029995500} \approx \underline{0.03}$$

$\boxed{\text{ASIDE}}$ $\dfrac{\sin(x)}{x} \approx 1$

AND GETS CLOSER AND CLOSER AS $x \to 0$

RECALL $\displaystyle\lim_{x \to 0} \frac{\sin(x)}{x} = 1$!

Example:

Using tangent line approximation estimate the value of $\sqrt[3]{8.5}$.

$$f(x) = \sqrt[3]{x} = x^{1/3}$$

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

$$f(8) = 8^{1/3} = 2$$

$$f'(8) = \frac{1}{3(8)^{2/3}} = \frac{1}{12}$$

$$y = \frac{1}{12}(x-8) + 2$$

$$L(x) = \frac{1}{12}(x-8) + 2$$

$$\boxed{\sqrt[3]{x} \approx \frac{1}{12}(x-8) + 2}$$

IF $x \approx 8$

3. ~~Using tangent line approximation estimate the value of $\sqrt[3]{8.5}$.~~

THUS

$$\underbrace{\sqrt[3]{8.5}}_{2.040827551} \approx \underbrace{\frac{1}{12}(8.5-8)}_{} + 2$$

$$\underbrace{\frac{1}{24} + 2}$$

$$2.041\bar{6}$$

ASIDE "Differentials"

NOTE $y = \frac{1}{12}(x-8) + 2$

CAN BE WRITTEN AS $\underbrace{y-2} = \frac{1}{12}\underbrace{(x-8)}$

WE SOMETIMES WRITE $\underbrace{dy = y-2}_{\text{change in } y \text{ on tangent line}}$ AND $\underbrace{dx = x-8}_{\text{change in } x \text{ on tangent line}}$

So $dy = \frac{1}{12}dx$

THIS IS JUST ANOTHER WAY TO SAY THE SAME THING!

Example: (Newton's Method)

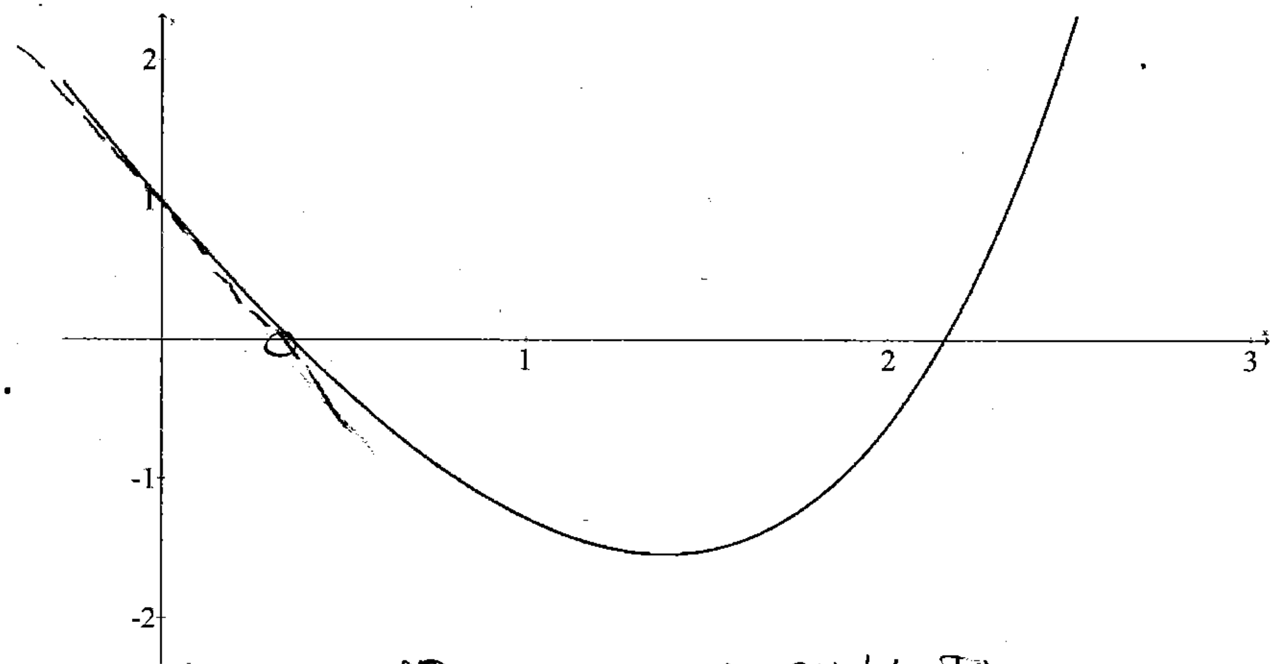
Consider $f(x) = e^x - 4x$.

From the graph, note that

$$e^x - 4x = 0$$

has one solution "near" $x = 0$.

Find the linearization at x=0, and use it to approximation this solution.



$f'(x) = e^x - 4$

$f(0) = e^0 - 4(0) = 1$

$f'(0) = e^0 - 4 = 1 - 4 = -3$

$y = -3(x-0) + 1 = -3x + 1$

$\Rightarrow e^x - 4x \approx -3x + 1$ for $x \approx 0$

WE APPROXIMATE A SOL'N TO

$\underbrace{e^x - 4x}$ $= 0$ BY REPLACING

$-3x + 1 \approx 0$

$\Rightarrow x = 1/3 = 0.\overline{3}$

("ACTUAL" VALUE $= 0.35740296$)

ASIDE!

IN NEWTON'S METHOD, WE USE $x = 1/3$ AND REPEAT THE PROCESS TO GET A BETTER APPROXIMATION

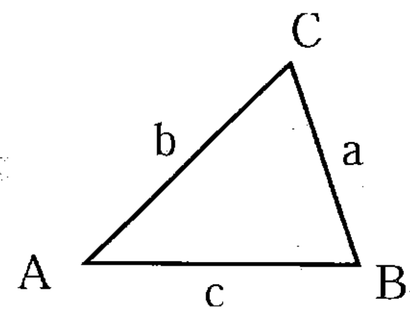
$\Rightarrow y = f'(1/3)(x - 1/3) + f(1/3) \overset{?}{=} 0$

$\Rightarrow x = \frac{1}{3} - \frac{f(1/3)}{f'(1/3)} \approx 0.357246$

AND AGAIN, AND AGAIN, UNTIL ANSWER IS PRECISE ENOUGH

Some Homework Hints:

Problem 10: Suppose that *a* and *b* are pieces of metal which are hinged at *C*.



By the ``law of sines,'' you *always* have:

$$\frac{b}{a} = \frac{\sin(B)}{\sin(A)}$$

At first: angle A is $\pi/4$ radians= 45^o and
angle B is $\pi/3$ radians = 60^o.

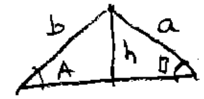
You then widen A to 46^o, without changing the sides a and b.

The question asks you to use the linear approximation to estimate the new angle B.

$$46^\circ = \underbrace{46 \frac{\pi}{180}} \text{ RADIANS}$$

ASIDE

PROOF



$\sin(A) = \frac{h}{b}$ AND $\sin(B) = \frac{h}{a}$

$\Rightarrow h = b\sin(A)$ AND $h = a\sin(B)$

$\Rightarrow b\sin(A) = a\sin(B) \Rightarrow \frac{b}{a} = \frac{\sin(B)}{\sin(A)}$ //

CONSTANTS

$b\sin(A) = a\sin(B)$

WE WANT THE TANGENT LINE $B = \square(A - \square) + \square$

?? dB/dA ?? A_0 $\pi/4$?? B_0 $\pi/3$

$$b\cos(A) = a\cos(B)\frac{dB}{dA}$$

$$\Rightarrow \frac{dB}{dA} = \frac{b\cos(A)}{a\cos(B)}$$

– At $B = \pi/3$ and $A = \pi/4$, WE HAVE

$$\frac{b}{a} = \frac{\sin(\pi/3)}{\sin(\pi/4)} = \frac{\sqrt{3}/2}{\sqrt{2}/2} = \sqrt{\frac{3}{2}}$$

AND $$\frac{dB}{dA} = \sqrt{\frac{3}{2}}\,\frac{\cos(\pi/4)}{\cos(\pi/3)} = \sqrt{\frac{3}{2}}\,\frac{\sqrt{2}/2}{1/2} = \sqrt{3}$$

$$\Rightarrow B = \sqrt{3}\,(A - \pi/4) + \pi/3$$